Title: Astronomy and the Wave Formula -- Mathematical Variation

Brief Overview:

In astronomical investigations, viewing the universe is dependent on electromagnetic radiation from emitting bodies reaching instruments on earth by means of waves. A conceptual understanding of the relationships among velocity, frequency, and wavelength is used as the basis for understanding direct and indirect variation and their graphical representations. The wave equation that results when all three variables are combined becomes the basis for problem solving and computing values for charting a spectrum to display the various electromagnetic emissions from the universe. Enrichment activities extend the lesson on variation to an equation containing a quantity squared.

Links to NCTM 2000 Standards:

- **Mathematics as Problem Solving**
  Once derived, the wave equation is used to produce a table of values which represent the different types of electromagnetic radiation (radio waves, microwaves, infrared, visible light, ultraviolet, X-rays, and gamma rays), using the speed of light as a constant velocity ($c= 3.0 \times 10^8$ m/s or 186,000 miles/sec). Problems can now be solved either by reasoning using conceptual understanding of the variations between variables or algebraically using the equation.

- **Mathematics as Reasoning and Proof**
  The concepts of direct and indirect variation are arrived at through reasoning from hands-on experience with whole waves as represented by shaped pipe cleaners. By seeing that shorter wavelengths require many more waves to fill a given distance, students construct for themselves the concept of an inverse or indirect relationship. The pipe cleaners are also used to reason direct variation between distance per second and wavelength plus direct velocity between velocity and frequency.

  Starting with proportionality expressions, an equation for waves is derived and proved by concrete demonstration with the constructed pipe cleaner waves and by use of appropriate units. Velocity and wavelength can be manipulated both in metric and English units. Frequency is always waves per second or cycles per second (hertz).

- **Mathematics as Representation**
  The table of electromagnetic wavelength and frequency variables is represented in a chart in either increasing or decreasing version, thus producing a spectrum to be compared with one from a text or poster. The relationship between frequency and wavelength should be graphed for a visual representation of indirect variation. Using constant frequency, direct variation between velocity and different wavelengths of visible light as when it passes through different media also can be graphed.

- **Mathematics as Communication**
  Research on the uses of the different wavelengths or frequencies and the energies they carry can be done on the Internet or through other resources. A written summary of the findings along with a description of the reasoning process undertaken to arrive at the equation will help students to communicate and demonstrate their understanding of the lesson.

  The repeat of the process for variables involving a quantity squared can be used for both assessment and extension purposes. The terms “parabola” and “quadratic” need not be introduced if new to the students since the emphasis is on understanding variation, starting from hands-on experience and resulting in graphs, charts, and an equation.

- **Mathematics as Connections**
  The equations and calculations will link directly to astronomical investigations.

- **Number and Operation**
  The operations of multiplying and dividing are needed for the wave equation and transforming it.

- **Patterns, Functions, and Algebra**
  Various graphs are produced depending on which variable is chosen to be the dependent variable (velocity, wavelength, or frequency) producing three functions. The pattern of increasing frequencies (decreasing wavelengths) produces the electromagnetic spectrum.

- **Measurement**
  Students must measure distances accurately in order to produce trustworthy replicas of waves. Distance and time measurements are required to determine velocities.

Links to Maryland High School Mathematics Core Learning Goals:

Functions and Algebra

- **1.1.1**
  The student will recognize, describe, and extend patterns and functional relationships that are expressed numerically, algebraically, and geometrically.

- **1.1.2**
  The student will represent patterns and functional relationships in a table, as a graph, and/or by mathematical expression.

- **1.1.3**
  The student will add, subtract, multiply, and divide algebraic expressions.

- **1.2.5**
  The student will apply formulas and use matrices to solve real-world problems.

Geometry, Measurement, and Reasoning

- **2.2.3**
  The student will identify and use inductive and deductive reasoning.

- **2.3.2**
  The student will use techniques of measurement and will estimate, calculate, and compare perimeter, circumference, area, volume, and surface area of two-and-three-dimensional figures and their parts. The results will be expressed with appropriate precision.

Data Analysis and Probability

- **3.1.1**
  The student will design and/or conduct an investigation that uses statistical methods to analyze data and communicate results.

- **3.1.2**
  The student will use the measures of central tendency and variability to make informed conclusions.

- **3.2.1**
  The student will make informed decisions and predictions based upon the results of simulations and data from research.

Links to National Science Education Standards:

- **Unifying Concepts and Processes**
  Energy and its transformation are processes of reasoning and algebraic operations with graphic representation.

- **Science as Inquiry**
  What is known about the universe has come to us through inquiry and investigation through the collection of data brought to us through electromagnetic radiation.

- **Physical Science**
  This lesson emphasizes energy transmission and the nature of electromagnetic radiation.

- **Earth and Space Science**
  This mathematical lesson relates to astronomy and the investigation of space.

Links to Maryland High School Science Core Learning Goals:

- **Concepts of Earth/Space Science**
  Data from the universe reaches us through emission of radiation. Appropriate instruments capture different frequencies of electromagnetic radiation. Research requires long term collection of data and careful analysis using mathematics.

- **Concepts of Physics**
  This learning unit encompasses: principles of energy transmission through waves; wave theory and the electromagnetic spectrum; and refraction and the role of varying wavelengths in producing the visible light spectrum.

Grade/Level:

Middle School (post Algebra I), High School (with extensions)

Duration/Length:

Five 45 minute periods or two to three double periods.

Prerequisite Knowledge:

Students should have working knowledge of the following skills:

- Using the metric system to represent small and large measurements
- Ability to use algebraic operations to solve equations
- Ability to compute a table of values by substitution into an equation
- Graphing on a Cartesian coordinate system.

Student Outcomes:

Students will:

- understand the sources of wave energies both from the universe and on earth.
- be able to illustrate the range of electromagnetic radiation by a spectrum using frequency and wavelength and the categories of energy emissions.
- understand conceptually the direct and indirect variations among velocity, frequency, and wavelength.

- be able to recognize variation (direct or indirect) by analysis of equations and graphs.
- be able to produce graphical representations of direct and indirect relationships.
- recognize that the graph of data can lead to the mathematical equation that describes the relationships among variables.

Materials/Resources/Printed Materials:

- Super slinky
- 12 inch pipe cleaners
- Stop watch
- Meter sticks or yard sticks
- Calculators
- Illustration of full spectrum for overhead or a poster-sized chart
- Internet access or texts as resources for research

Development/Procedures:

- **Step One: Electromagnetic Radiation -- Velocity, Waves, and Frequency**

  Discuss the transformation of energy in the sun from chemical (hydrogen) to vibrational energy to the emission of electromagnetic radiation. (Can be simple or complex). Discuss the types of radiation but postpone use of a prepared spectrum so that students can draw their own from data. Use hands-on activities such as a climbing rope, slinky, pond, or a row of people to discuss transportation of vibrational energy by waves. Have students *construct* in their own thinking concepts of velocity, frequency, and wavelength and other terminology for parts of a wave. Use Worksheet A.

  Questions for Consideration:
  What are different forms of energy?
  What is the difference between mass and energy?
  Is there an equation that relates the two quantities?
  How is energy carried across the universe?
  How do we utilize all the various forms of radiation whether generated by the sun or produced on earth?
  In a people-wave at the ballpark, what travels? How do the people "vibrate" to produce the wave? (transverse)
  Is there another way that particles can move to transmit energy?

- **Step Two: Mathematical Variation between Velocity, Wavelength, and Frequency**

  Develop a conceptual understanding of the relationships between velocity and frequency, velocity and wavelength, frequency and wavelength using pipe cleaners to form complete waves of varying wavelengths and laying waves within a given distance to demonstrate frequency. Use Worksheet B. Using proportionality expressions, show how the quantities are placed in an equation and arrive at the wave equation:

  velocity = frequency x wavelength

  Questions for consideration:
  If you keep the same wavelength but send more waves per second, does the wave travel faster or slower?
  If you keep frequency the same, but make the wavelength shorter, does the wave go faster or slower?

If the speed stays the same (keep distance constant) and wavelength increases, what happens to the frequency?

- **Step Three: Computation and Graphing**

Assign constant values to one variable at a time and have students compute three tables of values relating pairs of variables. Plot three graphs, compare with the positions of the variables within the wave equation, and assign the terms direct and indirect (or inverse) to the graphs.
Graph 1: If wavelength is 10 cm, compute a table and graph velocity versus frequency.
Graph 2: If frequency is 5 waves/second (5 hertz), graph velocity versus wavelength.
Graph 3: If velocity is 60 m/s, graph frequency versus wavelength.

Questions for consideration:
What does the graph of any direct relationship look like? indirect?
How can you determine relationships from the positions of variables in an equation?

- **Step Four: Algebraic Operations and Problem Solving**

Using the form v/f = wavelength, have students solve for velocity, and then frequency to practice transforming literal equations. Solving for a variable in the denominator position often causes difficulty for students.
Use the table (see Worksheet C) of types of electromagnetic radiation from the sun and the speed of light through a vacuum (3.0×10^8 m/s) to solve for the missing data.
Examine the data to see whether the relationship between frequency and wavelength verifies what was conceptually decided earlier.
Using the data from the computed table, have students construct their own charts showing scales for frequency and wavelength. Compare with a standard spectrum.

- **Step Five: Information Search for Utilization of Radiation**

Students research the applications of the various frequencies of electromagnetic radiation in medicine, industry, and science to be incorporated in the chart.
Enrichment and extension activities are investigated.

Assessment:

Students will complete the attached Assessment Quiz.

Rubric:

4: Students who confidently answer 9-12 questions correctly.

3: Students who answer 7-10 questions correctly.

2: Students who answer 4-7 questions correctly.

1: Students who confidently answer at least one question.

0: Students who choose not to participate.

Extension/Follow Up:

- Repeat the process of developing conceptually the relationships among variables using the $a = v^2/r$ equation for circular motion. Students do not have to understand its derivation but will use it to examine direct and indirect variation when one variable is squared. This extends the process of graphing to result in a parabola and then a linear relationship by using v^2 on the horizontal coordinate. Students should apply the same steps of computing a table, graphing, and verifying direct or indirect relationships in accord with their prior experience.
  At the high school level, a class that understands vectors can derive the circular acceleration equation using similar triangles and vectors.
  For hands-on experience, use the motion of an object being swung in a circle. Calculate speed and acceleration from circumference, time, and radius.

  Consider questions, such as:
  Is velocity constant? Speed? Acceleration?

- Students can write “papers” to demonstrate their understanding of variation using the circular motion equation ($a = v^2/r$). Applications to orbiting bodies and the placement of planets within a solar system should be included. If desired, universal gravitation as the force that counters circular motion can be introduced with further mathematical manipulation of equations ($g = F/m = a = v^2/r$).

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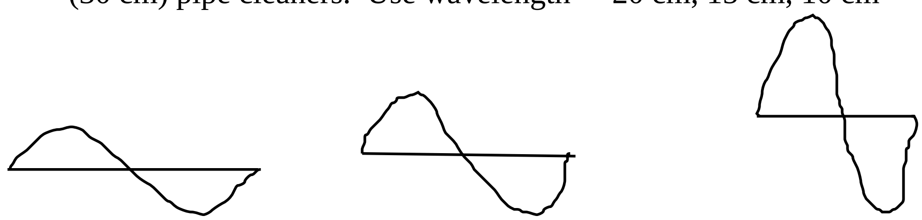
Worksheet A

Title: Hands-On Waves

Objective: To have students construct for themselves the concepts of indirect and direct variation using the wave formula: $v = f\lambda$.

Directions:
(At this time the students understand the term and units for velocity (m/s), wavelength (m), and frequency (waves/s or cycles/s or hertz).

Step 1. Instruct each student or group of students to form complete waves from 12 inch (30 cm) pipe cleaners. Use wavelength = 20 cm, 15 cm, 10 cm



20 cm. 15 cm. 10 cm.

Note: If pipe cleaners are not cut, the amplitude will necessarily increase as the wavelengths decrease. This can be incorporated into the lesson but point out that this relationship is simply a peripheral characteristic. If the amplitude is to be the same, it would have to be designated and the pipe cleaners cut. We recommend just working with the different amplitudes without emphasizing this variation.

Step 2. Have students mark off on the floor or lab table with masking tape or chalk, distances of 1.5 meters, 1.0 meters, .5 meters. These distances represent the distances that will be traveled by various waves in one second (velocities).

Step 3. Now have students investigate these situations:

1. If 1.5 meters are to be traveled in one second, how many of each size wave are required? What term is given to the number you have determined?
Do this for each distance marked on the lab table and construct a table to record your data. (The teacher can facilitate this step but students should be encouraged to design the table themselves.)

2. Ask the students, if you had 13 waves that are 5 cm long pass by in one sec, how fast are they moving? What about seven 15 cm waves? How fast would eight waves that are 10 cm long travel? Record your data in a table.

3. Have the students demonstrate the following with their materials, if ten waves moved three quarters of a meter in a second, how long would each wave be?

4. Have the students make up one problem and demonstrate a hands-on solution to prove their answer.

Worksheet B

Title: Derivation of the Wave Equation

Objective: To have students reason to a combined relationship among velocity, frequency, and wavelength.

Have the students consider the following reasoning questions:

1. If velocity stays constant, what happens to frequency as the wavelength changes?

As wavelength increases, frequency decreases.
As wavelength decreases, frequency increases. or

$f \propto 1 / \lambda$ (indirect variation)

2. If the frequency stays the same, what happens to the velocity as the wavelength changes?

As wavelength increases, velocity increases.
As wavelength decreases, velocity decreases. or

$v \propto \lambda$ (direct variation)

3. If wavelength stays does not change, what happens to velocity as the number of waves that pass by in one second varies?

$v \propto f$ (direct variation)

Putting these three variables into one relationship so that the variations stated above remain true, yields

$$\boxed{v = f \lambda}$$

or the wave equation which can now be used to compute values and solve problems.

Worksheet C

Title: A Chart of Electromagnetic Radiation

Objective: To have students use the wave equation to generate data which will then be represented and summarized in an illustrated chart.

Instructions:

Complete this table using the wave formula. For velocity use 3.0×10^8 m/s or 186,000 miles/s.

Type of Radiation	Wavelength	Frequency
infrared	10^{-2} cm	
X rays		10^{18} hertz(hz)
microwaves		10^{11} hz
green	600×10^{-6}	
gamma rays	10^{-12} cm	
radio waves		10^{8} hz
yellow	500×10^{-6}	
ultraviolet		10^{16} hz

From your data, design a chart with the lowest frequency on the left, spacing the various quantities proportionally.

Research the uses of each type and place this on your chart (spectrum). Use text or illustrations or a combination of both.

What instruments will detect these various types of radiation from space? Which have been sent into space as satellites or probes? If possible, obtain the photographic results of each of these means of obtaining data from space.

Assessment Quiz

Instructions: Read each statement carefully and decide if it is Always true (A), Sometimes true (S), or Never true (N).

1. _______ In the wave formula, wavelength varies directly with frequency.

2. _______ Different wavelengths of radiation from the sun travel at different speeds.

3. _______ If green light passes into water, the frequency changes.

4. _______ When frequency does not change, the velocity decreases as the wavelength gets longer.

5. _______ If the frequency changes, the velocity increases.

6. _______ Two quantities that are directly related can be written as numerators on the opposite sides of an equation.

7. _______ Two quantities that are indirectly related are written as numerators on one side of the equation.

8. _______ Radiation transmits particles of matter from the Sun to the Earth.

9. _______ A graphic representation of all of the frequencies of electromagnetic radiation is called the light curve.

10. _______ $f / v = \lambda$

11. _______ The graph of a direct relationship is a parabola.

12. _______ Sound is transmitted through longitudinal waves.

Assessment Quiz Key

Instructions: Read each statement carefully and decide if it is Always true (A), Sometimes true (S), or Never true (N).

1. ___N___ In the wave formula, wavelength varies directly with frequency.

2. ___N___ Different wavelengths of radiation from the Sun travel at different speeds.

3. ___N___ If green light passes into water, the frequency changes.

4. ___A___ When frequency does not change, the velocity decreases as the wavelength gets longer.

5. ___S___ If the frequency changes, the velocity increases.

6. ___A___ Two quantities that are directly related can be written as numerators on the opposite sides of an equation.

7. ___S___ Two quantities that are indirectly related are written as numerators on one side of the equation.

8. ___N___ Radiation transmits particles of matter from the Sun to the Earth.

9. ___N___ A graphic representation of all of the frequencies of electromagnetic radiation is called the light curve.

10. ___N___ $f / v = \lambda$

11. ___A___ The graph of a direct relationship is a straight line.

12. ___A___ Sound is transmitted through longitudinal waves.